

Discrete energy spectrum of Hawking radiation from Schwarzschild surfaces

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Abstract

We analyze the allowed energy levels of Hawking radiation from Schwarzschild surfaces in space-times with large extra dimensions. From the requirement that the wave functions associated with these particles be single-valued and obey radial boundary conditions, we derive an upper bound of their discrete energy spectrum as a function of the number of large extra dimensions and the mass of the black hole. Furthermore, we investigate the spacing of the energy levels.

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Space-times with large extra dimensions (LXD) have been proposed [1,2] to explain or at least reformulate the hierarchy between the electroweak scale $M_w \approx 10^3$ GeV and the Planck scale $M_{Pl} \approx 10^{19}$ GeV in four dimensions. In these brane world scenarios the hierarchy is either generated through the volume of the LXD [1] or through an exponential warp factor [2] taking into account the energy density on the brane. A common feature of these scenarios is that all standard model particles including the gauge degrees of freedom are localized on a 3-brane whereas gravity propagates in both the compact LXD and the non-compact dimensions. As a consequence the fundamental scale of gravity M_f could be as low as M_w in these space-times.

The experimental bound on M_f from the absence of missing energy signatures is $M_f \geq 800$ GeV [3–5]. Therefore, LXD may be discovered at Tevatron [6] and further explored with future colliders like LHC, TESLA or CLIC. One possible phenomenon could be the formation of micro black holes in high energy collider experiments [7] or atmospheric ultra high energy cosmic rays [8]. The production of micro black holes in the final state of a high energy collision could be observed due to a sharp cut-off [9] in $\sigma(p\bar{p} \rightarrow \text{Jet} + X)(P_\perp)$ at Tevatron or $\sigma(pp \rightarrow \text{Jet} + X)(P_\perp)$ at LHC.

One of the most remarkable relationships in theoretical physics is that between classical black hole physics and the ordinary laws of thermodynamics [10]. This relation is a striking mathematical analogy and suggests a physical connection. The quantity in classical black hole physics which plays the role mathematically analogous to the total Energy U is the mass M_{bh} of the black hole, which is physically the total energy of the black hole in general relativity. The quantity analogous to the temperature is the surface gravity κ . In classical black hole physics κ has nothing to do with the physical temperature of a black hole, which is absolute zero. However, the analysis of the behavior of a quantum field results in the Hawking effect [11]: A black hole will radiate exactly like a black body with temperature $T_{bh} = \kappa/2\pi$. The thermal spectrum of radiated quanta can be calculated by noting that the physical entropy S of a black hole is proportional to its Schwarzschild surface area. For a spherically symmetric solution in $D = 4 + d$ dimensions $S \sim M_f^{2+d} A_{(2+d)} R_{bh}^{2+d}$, where $A_{(2+d)}$ denotes the $2 + d$ -dimensional surface of a $3 + d$ -dimensional unit sphere and the horizon radius is given by [12]

$$R_{bh}^{1+d} = \frac{4}{2+d} \frac{A_{(2)}}{A_{(2+d)}} \left(\frac{1}{M_f} \right)^{1+d} \frac{M_{bh}}{M_f}. \quad (1)$$

In the literature [13] it was argued that the thermal spectrum of radiated quanta should be calculated using the micro-canonical ensemble. In the following we formulate two essential conditions which restrict this spectrum due to finite size effects. It is important to note that no special ensemble is considered in deriving the main result Equation (7).

Condition 1 : The energy ω of the radiated quanta has an upper bound provided by the mass M_{bh} of the black hole

$$\omega \leq M_{bh}/2, \quad (2)$$

which is mandatory due to energy-momentum conservation. This condition is equivalent to the statement that the de Broglie wavelength of the quantum formed on the Schwarzschild

surface needs to be larger than the Compton wave length of the micro black hole.

Condition 2 :

- (a) For every de Broglie wavelength λ associated with a particle on the Schwarzschild surface exists a positive integer n with

$$n\lambda = R_{\text{bh}} , \quad (3)$$

which implies $\lambda \leq R_{\text{bh}}$.

- (b) The radial wave function F has to vanish on the Schwarzschild surface, i. e.

$$F(r) = 0 \quad \text{for} \quad r \sim R_{\text{bh}} . \quad (4)$$

Condition (a) is necessary and sufficient in order to allow for periodic boundary conditions so that the wave-function of a particle localized on the Schwarzschild surface is single-valued. If the wavelength is larger than the horizon radius destructive interference will suppress the amplitude for such a particle state. Condition (b) ensures that the wave function of the particle does not penetrate beyond the horizon. Conditions (a) and (b) are non-trivial boundary conditions that result in a geometrical quantization of all components of the particle momentum \vec{k} .

Application to black holes :

Ignoring numerical prefactors we obtain from *Condition 1* and *Condition 2* the general relation

$$\frac{1}{M_{\text{f}}} \left(\frac{M_{\text{bh}}}{M_{\text{f}}} \right)^{\frac{1}{1+d}} \geq \lambda \geq \frac{1}{M_{\text{bh}}} \quad (5)$$

implying $M_{\text{bh}} \geq M_{\text{f}}$. For $d = 0$ this relation is easily satisfied for astrophysical black holes, thereby allowing a large range for λ . However, the above stated conditions imply a general self-consistency relation for arbitrary d .

Modification of black body spectrum :

The remarkable conclusion known as Hawking effect is that, at late times, a Schwarzschild black hole formed by gravitational collapse radiates precisely as a thermal black body at temperature $T_{\text{bh}} = (1 + d)/(A_{(2)}R_{\text{bh}})$. The number of momentum modes that can be thermally populated is roughly $R_{\text{bh}}^3 k^2 dk$. If this number changes continuously in momentum space, the amount of energy radiated away at a given T_{bh} is an integral of the Planck spectrum

$$u(\omega, T_{\text{bh}}) = \sum \frac{g}{2\pi^2} k^2 f(\omega/T_{\text{bh}}) \omega \quad (6)$$

over the energy. Here, the sum is over all possible particle species, g counts the effective degrees of freedom and f denotes the Bose-Einstein or the Fermi-Dirac distribution respectively for every particle species. Planck's radiation law is reliable for $M_{\text{bh}}/T_{\text{bh}} \gg 1$, where in our case $M_{\text{bh}}/T_{\text{bh}} \approx (M_{\text{bh}}/M_{\text{f}})^{(2+d)/(1+d)}$.

In the following we examine when the assumption of continuous variation of allowed momentum values really holds. Let us consider a Schwarzschild sphere centered in a cube

with side length $2R_{\text{bh}}$. The energy spectrum of modes on the Schwarzschild surface can be approximated by the spectrum of modes inside the cube. Then, *Condition 2* is modified by $R_{\text{bh}} \rightarrow 2R_{\text{bh}}$. As a consequence, the relation $k = \pi n/R_{\text{bh}}$ between the momentum k of the particle (with mass m), the positive integer n and the horizon radius follows. With *Condition 1* we find our main result

$$n(d, M_{\text{bh}}) \leq (4\pi^3)^{-\frac{1}{2}} \left(1 - \frac{4m^2}{M_{\text{bh}}^2}\right)^{\frac{1}{2}} \left(\frac{8}{2+d} \Gamma\left(\frac{3+d}{2}\right)\right)^{\frac{1}{1+d}} \left(\frac{M_{\text{bh}}}{M_{\text{f}}}\right)^{\frac{2+d}{1+d}}. \quad (7)$$

Again, for $d = 0$ and macroscopic black holes we have typically $n \sim 10^{60}$ and therefore a continuous thermal spectrum of radiated quanta. On the other hand, for $M_{\text{bh}} \sim M_{\text{f}}$, n is of order one and k is quantized due to the small size of the Schwarzschild surface. *Condition 2* then restricts the phase space in the infrared and allows no modes with $k \ll T$. Note that in deriving (7) we did not specify any ensemble.

As a consequence, for black holes with $M_{\text{bh}} \sim M_{\text{f}}$ only a discrete energy spectrum is possible with finite spacing between the levels. Instead of integrating the Planck spectrum over the energy we have a sum over the discrete levels

$$\sum_{n=1}^{[n \leq n_{\text{max}}]} u(\omega_n, T_{\text{bh}}). \quad (8)$$

Here, n_{max} is the right hand side of (7) and $[\dots]$ denote the largest integer part of the argument in brackets. The energy spacing between a mode with momentum k and a mode with momentum $k + dk$ is approximately $1/R_{\text{bh}}$. Since (7) is general, similar conclusions should hold for the micro-canonical ensemble, which we will discuss in a forthcoming publication. Here we illustrate our results using the grand canonical ensemble, where

$$\frac{dM_{\text{bh}}}{dt} = -A_{(2)} R_{\text{bh}}^2 \sum_{n=1}^{[n \leq n_{\text{max}}]} u(\omega_n, T_{\text{bh}}) \quad (9)$$

for standard model fields [14], taking into account finite spacing between the energy levels. For a black hole with $M_{\text{bh}} = 15M_{\text{f}}$ and $d = 2$ we find $n_{\text{max}} \approx 5$. However, the spectrum is peaked for $\omega \approx M_{\text{f}}$, i.e. for the lowest energy level accessible for a particle on the Schwarzschild surface. Nearly all energy goes in the lowest possible energy mode, as shown in Fig. 1. Neglecting modes with higher energies results in a relative error of the order of one percent. For $d = 3 - 7$ the situation is similar, though n_{max} is decreasing and the maximum relative error is of the order of ten percent. With increasing ω the d -dependence becomes more important. For example, for the second energy level we find $\omega \approx 2M_{\text{f}}$ for $d = 2$ and $\omega \approx 4M_{\text{f}}$ for $d = 7$.

In conclusion, we have shown that the energy spectrum of particles emitted by M_{f} -scale black holes is quantized due to non-trivial boundary conditions. For the lowest lying energy level accessible for particles on the Schwarzschild surface we find $\omega \approx M_{\text{f}}$. Nearly all energy is radiated off the Schwarzschild surface in modes with this lowest energy. This may have dramatic consequences. Depending on the detailed prescription for geometrical quantization, the black holes on the M_{f} -scale may not evaporate at all via Hawking radiation.

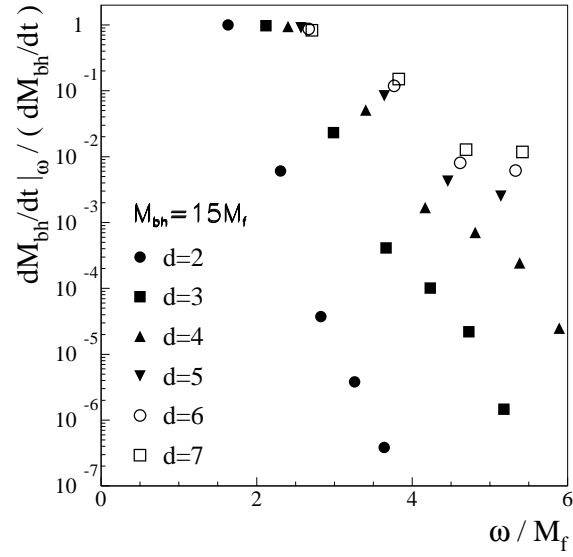


FIG. 1. The power emitted via standard model particles in a certain frequency mode normalized to (8) as a function of ω/M_f for $d \in \{2, 3, 4, 5, 6, 7\}$.

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